(6.) The foot is naturally cold or artificially cooled.

These are, I think, the reasons for the facility with which the magicians perform their "fire-walk," and I must say that it is a smart piece of jugglery or "savage magic," and not by any means an inexplicable mystery.

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(j.) Christchurch *Press*, 16th July, 1902, p. 6.
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ART. XIV.—*The Adjustment of Triangulation by Least Squares.*

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IT is proposed in the following paper to select examples of the ordinary methods of adjusting triangulation as practised in New Zealand and to apply to them the least-square adjustment, so as to compare the relative results obtained, and to show by actual examples that this method of adjustment alters the observed angles less than any other method. It will also be shown that the least-square adjustment can be simply and readily applied to most cases that occur in practice.

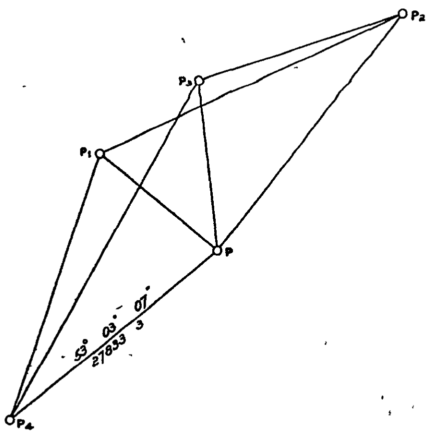
It is hoped that this treatment of the subject will be of use to the practical computer, and that it will enable him to see the advantages of the least-square adjustment by comparing its results with those usually obtained.

To make the treatment as simple as possible it will be assumed that all the angles are equally well observed.

Example No. 1.—The Adjustment of Four Plane Triangles.

For the purposes of comparison with the usual method of adjusting triangulation (in which one-third of the triangular error is applied to each angle of the triangle) an easy example is selected embracing the adjustment of four plane triangles.

The side P_4 P in the figure is a side of the existing triangulation, and is to be adopted as correct both in bearing and length. From this side it is desired to extend the triangulation so as to include the points P_1 P_3 and P_2. The angles are all observed, and are shown in column No. 2 of the schedule. No observation was possible between P_1 and P_3.



I. *Adjustment as Four Separate Triangles.*

The angles in each triangle are adjusted by applying to each angle one-third of the triangular error of the triangle.

The adjustment is shown in the schedule in columns 1 to 4.

In column 1 the names of the angles are entered as follows (see figure):—

$$A_1 = P\ P_1\ P_4 \qquad A_2 = P\ P_2\ P_1$$
$$B_1 = P_1\ P_4\ P \qquad B_2 = P_2\ P_1\ P$$
$$C_1 = P_4\ P\ P_1 \qquad C_2 = P_1\ P\ P_2$$

$$A_3 = P\ P_3\ P_2 \qquad A_4 = P\ P_4\ P_3$$
$$B_3 = P_3\ P_2\ P \qquad B_4 = P_4\ P_3\ P$$
$$C_3 = P_2\ P\ P_3 \qquad C_4 = P_3\ P\ P_4$$

Column No. 2 contains the observed angles.

In column No. 3 one-third of the triangular error of each triangle is applied to each angle.

Column No. 4 gives the sums of the angles (seconds only) of columns Nos. 2 and 3.

With these angles (from column 4) the triangles are calculated, and the following results are obtained:—

First Pair of Triangles—P_1 P_4 P and P_2 P_1 P.

Side.	Bearing.	Distance. Links.	Remarks.
P P_4	53° 03′ 07″	27833·3	Adopted as base.
P_4 P_1	20° 13′ 46·3″	29069·9	
P_1 P	130° 51′ 38·7″	16120·5	
P_1 P_2	67° 42′ 42·7″	34843·4	
P_2 P	220° 09′ 19·7″	31089·0	

Second Pair of Triangles—P_3 P_2 P and P_4 P_3 P.

Side.	Bearing.	Distance. Links.	Remarks.
P P_4	53° 03′ 07″	27833·3	Adopted as base.
P_4 P_3	30° 49′ 18·7″	40207·6	
P_3 P	174° 43′ 37·3″	17874·2	
P_3 P_2	74° 37′ 29·0″	22499·4	
P_2 P	220° 09′ 24·7″	31093·3	

Comparing the bearing and length of the side P_2 P as obtained from the two sets of triangles, we have—

220° 09′ 19·7″ 31089·0 links; and
220° 09′ 24·7″ 31093·3 „

giving differences of 5″ and 4·3 links.

The application of the ordinary adjustment, resulting as it does in these differences, is therefore very unsatisfactory, and the question arises as to whether it is desirable in this and in similar cases to adopt some further adjustment to the observed angles so as to eliminate the discrepancies shown above.

Before discussing the further adjustment it may be as well to remark that the ordinary procedure would be to adopt the mean values of the bearing and distance of P P_2. None of the other sides, however, would receive any correction; consequently if the calculation is repeated, using the mean value of P P_2 as base, an entirely different set of values will be obtained for all the other sides of the triangles.

As the need for further adjustment is obvious, the method of applying it will now be indicated.

II. *The Least-square Adjustment.*

The problem to be solved is: Given the observed angles of the four triangles, corrected as shown in I., by applying one-third of the error of each triangle to each angle, what further corrections must be made to these angles so as to eliminate the discrepancies found above?

It is evidently desirable that the corrections should be as small as possible so that no undue alterations are made to the angles: this condition is satisfied when the sum of the squares of the corrections is a minimum.

The application of this condition is shown on the schedule, and is briefly as follows:—

In column No. 5 the natural sines of the angles in column No. 4 are given.

If the sines in No. 5 were correct we should have—

$$\frac{\text{Sin } A_1 \sin A_2 \sin A_3 \sin A_4}{\text{Sin } B_1 \sin B_2 \sin B_3 \sin B_4} = 1.$$

This equation shows that the length of P P_2 calculated from P P_4 by the first pair of triangles should be the same as the length calculated by the second pair of triangles.

This is not usually the case, so put

$$\frac{\text{Sin } A_1 \sin A_2 \sin A_3 \sin A_4}{\text{Sin } B_1 \sin B_2 \sin B_3 \sin B_4} = 1 + \epsilon,$$

where the sines are taken from column No 5 and ϵ is n radians.

To convert ϵ into seconds multiply the value in radians by 206265 (= number of seconds in 1 radian).

The calculation is shown on the schedule, giving, in this particular example, $\epsilon = + 28''{\cdot}01$.

(NOTE.—Attention must be paid to the sign of ϵ.)

The other necessary condition is that the sum of the angles C_1 and C_2 should equal the sum of the angles of C_3 and C_4, or—

$$C_1 + C_2 = C_3 + C_4.$$

This is not usually the case, so put

$$C_1 + C_2 = C_3 + C_4 + \epsilon_0,$$

where the angles are taken from column No. 4 and ϵ_0 is in seconds. This gives $\epsilon_0 = -5''$ in this example (see schedule).

(NOTE.—Attention must be paid to the sign of ϵ_0.)

In column No. 6 the natural cotangents of the angles are inserted.

In column No. 7 twice the cotangents are entered.

$$\text{Let } \alpha_1 = \cot A_1 \qquad \alpha_2 = \cot A_2$$
$$\text{,, } \beta_1 = \cot B_1 \qquad \beta_2 = \cot B_2$$

and similarly for the other angles.

$$\text{Let } a_1 = 2\alpha_1 + \beta_1 \qquad a_2 = 2\alpha_2 + \beta_2$$
$$\text{,, } b_1 = -\alpha_1 - 2\beta_1 \qquad b_2 = -\alpha_2 - 2\beta_2$$
$$\text{,, } c_1 = -\alpha_1 + \beta_1 \qquad c_2 = -\alpha_2 + \beta_2$$

and similarly for a_3, b_3, c_3: a_4, b_4, c_4.

In column No. 8 the values of a_1, b_1, c_1, &c., are given, and a check is obtained by noting that $a_1 + b_1 + c_1 = o$.

Square all the values in column No. 8 and add them. This is done readily on the Brunsviga calculating-machine without any intermediate record, the result in this example being—

$$\Sigma (a^2 + b^2 + c^2) = + 126{\cdot}790.$$

Let $k = \frac{1}{6} \Sigma (a^2 + b^2 + c^2)$

" $h = c_1 + c_2 - c_3 - c_4$ (from column No. 8).

" $i =$ the number of triangles.

Next form the following equations :—

$$hP + 2iQ + \epsilon_0 = o$$
$$2kP + hQ + \epsilon = o$$

and solve them for P and Q.

With these values of P and Q calculate the corrections to the observed angles thus—

$$x_1 = a_1P - Q \qquad x_2 = a_2P - Q$$
$$y_1 = b_1P - Q \qquad y_2 = b_2P - Q$$
$$z_1 = c_1P + 2Q \qquad z_2 = c_2P + 2Q$$

$$x_3 = a_3P + Q \qquad x_4 = a_4P + Q$$
$$y_3 = b_3P + Q \qquad y_4 = b_4P + Q$$
$$z_3 = c_3P - 2Q \qquad z_4 = c_4P - 2Q$$

where x_1, y_1, z_1, &c., are the corrections in seconds, and the corrected angles are—

$$A_1 + x_1 \qquad A_2 + x_2 \qquad A_3 + x_3 \qquad A_4 + x_4$$
$$B_1 + y_1 \qquad B_2 + y_2 \qquad B_3 + y_3 \qquad B_4 + y_4$$
$$C_1 + z_1 \qquad C_2 + z_2 \qquad C_3 + z_3 \qquad C_4 + z_4$$

where the angles A_1, B_1, &c., are taken from column No. 4.

Columns 9, 10, and 11 show the calculation of the corrections.

Column No. 12 gives the final angles (seconds only), and is equal to column 4 + column 11.

Column No. 13 gives the natural sines of the angles in Column No. 12.

This completes the calculation of the least-square corrections.

In practice it is always desirable to check the results obtained, consequently the two following checks are applied :—

(α.) By forming the products of—

Sin A_1 sin A_2 sin A_3 sin A_4 (from column 13); and
Sin B_1 sin B_2 sin B_3 sin B_4 "

which are equal (see schedule).

(β.) By comparing the values of—

$C_1 + C_2$ (from column 12); and
$C_3 + C_4$ "

which are equal.

The triangles are now solved, using the angles from column 12, and the results are—

First Pair of Triangles.

Side.	Bearing.	Distance. Links.	Differences. Bearing.	Differences. Distance. Links.
$P\ P_4$	53° 03′ 07″	27833·3	...	...
$P_4\ P_1$	20° 13′ 44·6″	29070·1	+ 1·7″	− 0·2
$P_1\ P$	130° 51′ 39·0″	16120·8	− 0·3″	− 0·3
$P_1\ P_2$	67° 42′ 41·7″	34845·1	+ 1·0″	− 1·7
$P_2\ P$	220° 09′ 22·1″	31090·6	− 2·4″	− 1·6

Second Pair of Triangles.

Side.	Bearing.	Distance. Links.	Differences. Bearing.	Differences. Distance. Links.
$P\ P_4$	53° 03′ 07″	27833·3	...	...
$P_4\ P_3$	30° 49′ 22·2″	40206·6	− 3·5″	+ 1·0
$P_3\ P$	174° 43′ 36·9″	17873·0	+ 0·4″	+ 1·2
$P_3\ P_2$	74° 37′ 28·7″	22497·3	+ 0·3″	+ 2·1
$P_2\ P$	40° 09′ 22·1″	31090·6	+ 2·6″	+ 2·7

The columns headed "Differences" give the differences between the least-square values and the values obained in I.

A comparison of the values of P_2 P as calculated from each pair of triangles shows that the bearing and distance agree exactly.

The process of adjustment here described completely satisfies the geometrical conditions of the figure, and it does so by making the sums of the corrections the least possible.

For the theory of the adjustment reference must be made to any of the treatises on least squares. See in particular "Geodesy," by Colonel A. R. Clarke, C.B., Oxford, 1880, pp. 217–225. The method here outlined differs from that given in Clarke, inasmuch as the triangular error is applied before the condition equations are derived, thus lightening the subsequent work very considerably, and thereby lessening the risk of numerical slips.

This method also permits of comparison between the ordinary triangular adjustment and the least-square adjustment, as will be seen by comparing columns 3 and 11, where column 11 shows the additional corrections necessary to satisfy the geometrical conditions of the figure.

In this example the calculation has been carried to two decimal places of a second, not because the observations justify so much refinement, but to avoid an unequal distribution of the errors, as, for instance, would occur in distributing an error of 2″ among three angles. If this is done to the nearest second, then two angles would receive a correction of one second each and the third angle would remain unaltered. This would not have been consistent with the theory of the adjustment, which provides that exactly one-third of the triangular error must be applied to each angle.

The whole of the calculations have been done on the Brunsviga calculating-machine with ease, rapidity, and certainty.

SCHEDULE.

(1.) Angles.	(2.) Observed Angles.	(3.) Correction ⅓ Error of Δ.	(4) =(2)+(3). Corrected Angles.	(5) = Sines of (4). Sines of Corrected Angles.	(6.) Cot. of (2).	(7.) Twice Cot of (2).	(8.) a, b, c	(9) =(8)×P.	(10.) Q and 2 Q.	(11) =(9)+(10). Least-square Corrections to Angles.	(12) =(4)+(11) Final Angles.	(13.) Sines of (12).
	° ′ ″	″	″					″	″	″	″	
A_1	69 22 7	+0·67	7·67	0·9358678	$\alpha_1=+0·376$	+0·752	$a_1=+2·302$	−1·501	−0·560	$x_1=-2·061$	5·61	0·9358643
B_1	32 49 20	+0·66	20·66	0·5420369	$\beta_1=+1·550$	+3·100	$b_1=-3·476$	+2·266	−0·560	$y_1=+1·706$	22·37	0·5420438
C_1	77 48 31	+0·67	31·67	0·9774483	..	..	$c_1=+1·174$	−0·765	+1·120	$z_1=+0·355$	32·02	0·9774487
	179 59 58											
A_2	27 33 24	−1·00	23·00	0·4626214	$\alpha_2=+1·916$	+3 832	$a_2=+4·338$	−2·828	−0·560	$x_2=-3·388$	19·61	0·4626068
B_2	63 8 57	−1·00	56·00	0·8921832	$\beta_2=+0·506$	+1·012	$b_2=-2·928$	+1·909	−0·560	$y_2=+1·349$	57·35	0·8921862
C_2	89 17 42	−1·00	41·00	0·9999243	..	..	$c_2=-1·410$	+0 919	+1·120	$z_2=+2·039$	43·04	0·9999244
	180 0 3											
A_3	100 6 13	−4·67	8·33	0·9844961	$\alpha_3=-0·178$	−0·356	$a_3=+1·100$	−0·717	+0·560	$x_3=-0·157$	8·18	0·9844962
B_3	34 28 9	−4·66	4·34	0·5659440	$\beta_3=+1·456$	+2·912	$b_3=-2·734$	+1·783	+0·560	$y_3=+2·343$	6·67	0·5659534
C_3	45 25 52	−4·67	47·33	0 7123913	..	..	$c_3=+1·634$	−1·065	−1·120	$z_3=-2·185$	45·15	0·7123839
	180 0 14											
A_4	22 13 51	−2·67	48·33	0·3783270	$\alpha_4=+2·446$	+4·892	$a_4=+6·263$	−4·083	+0·560	$x_4=-3·523$	44·81	0·3783112
B_4	36 5 44	−2·67	41·33	0·5891232	$\beta_4=+1·371$	+2·742	$b_4=-5·188$	+3·383	+0·560	$y_4=+3·943$	45·28	0·5891387
C_4	121 40 33	−2·66	30·34	0·8510394	..	..	$c_4=-1·075$	+0·701	−1·120	$z_4=-0·419$	29·91	0·8510405
	180 0 8						$\Sigma(a^2+b^2+c^2) = +126·790$					

$$\frac{\text{Sin } A_1 \text{ sin } A_2 \text{ sin } A_3 \text{ sin } A_4 \text{ (from column 5)}}{\text{Sin } B_1 \text{ sin } B_2 \text{ sin } B_3 \text{ sin } B_4 \quad \text{''}} = \frac{0.1612581}{0.1612362} = 1 + \epsilon.$$

$$\therefore 1 + \epsilon = 1.0001358.$$
$$\therefore \epsilon = + 0.0001358 \text{ radians}.$$
$$\therefore \epsilon = + 28''.01.$$

$C_1 + C_2 = C_3 + C_4 + \epsilon_0$ (from column 4).

$$\therefore 31''.67 + 41''.00 = 47''.33 + 30''.34 + \epsilon_0.$$
$$\therefore \epsilon_0 = - 5''.00.$$

$k = \frac{1}{6} \Sigma (a^2 + b^2 + c^2) = + 21.132$ (from column 8).
$h = c_1 + c_2 - c_3 - c_4 = - 0.795$ "
$i = 4.$

The equations for P and Q are—

$$hP + 2iQ + \epsilon_0 = O.$$
$$2kP + hQ + \epsilon = O.$$

Substitute the values of h, k, i, ϵ_0 and ϵ, and the equations become—

$$- 0.795 \text{ P} + 8 \text{ Q} - 5.00 = O.$$
$$+ 42.264 \text{ P} - 0.795 \text{ Q} + 28.01 = O.$$

Solving these equations for P and Q, we obtain—

$$P = - 0.652.$$
$$Q = + 0.560.$$

Checks on Final Angles.

Sin A_1 sin A_2 sin A_3 sin $A_4 = 0.1612457$ (from column 13).
Sin B_1 sin B_2 sin B_3 sin $B_4 = 0.1612457$ "

$C_1 + C_2 = 75''.06$ (from column 12).
$C_3 + C_4 = 75''.06$ "