V. "Transactions of the New Zealand Institute," vol. xxvii, p. 273.
W. "Westland: Geology of Hokitika Sheet, North Quadrangle," 1906, p. 13.
X. "Zoologist," 1871, vol. xxix.
Y. "Zoologist," 1881, p. 290.
Z. "Zoologist," 1883, p. 276.

EXPLANATION OF PLATE XV.

MAP OF THE SOUTH ISLAND OF NEW ZEALAND, SHOWING THE KEA'S DISTRIBUTION.

No. 1. Places where keas have been seen to attack sheep and authentic accounts have been sent in.
No. 2. Places where keas have been reported to have attacked sheep but no accounts have been sent in.
No. 3. Place where keas have been reported to have been seen.
No. 4. Capital towns of the provinces.

ART. XXIX.—*On Isogonal Transformations: Part I.*

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[*Read before the Philosophical Institute of Canterbury, 5th December, 1906.*]

1. "Two points P, P′, which are such that lines drawn from them to the summits of the triangle of reference are equally inclined to the bisectors of its angles are called isogonal conjugates with respect to the triangle."—*Casey.*

If the trilinear co-ordinates of P be $(\alpha\,\beta\,\gamma)$, those of P′ will be $\left(\frac{\kappa^2}{\alpha}\,\frac{\kappa^2}{\beta}\,\frac{\kappa^2}{\gamma}\right)$; but as in what follows trilinear ratios will be for the most part used, the co-ordinates of P′ will be $\left(\frac{1}{\alpha}\,\frac{1}{\beta}\,\frac{1}{\gamma}\right)$. If the co-ordinates of P′ be written $(\alpha'\beta'\gamma')$ we have $\alpha\alpha' = \beta\beta' = \gamma\gamma' =$ a constant: hence an isogonal transformation is a species of inversion, and in the following paper isogonal transformations will be described in the language of inversion.

The incentre and three excentres $(1 \pm 1 \pm 1)$ of the triangle of reference ABC are the only points which invert into themselves. The four points $(\alpha \pm \beta \pm \gamma)$ forming the vertices of a harmonic quadrangle invert into four points $\left(\frac{1}{\alpha} \pm \frac{1}{\beta} \pm \frac{1}{\gamma}\right)$ forming the summits of another harmonic quadrangle.

It may also be noticed that according as P is within or without the triangle ABC so is its inverse point P′ within or without that triangle.

2. The line whose equation is

$$l\alpha+m\beta+n\gamma=o$$

will invert into the conic having for equation

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$$

Also, any conic circumscribed to the triangle ABC will invert into a line: in particular the circumcircle of the triangle ABC will invert into the line at infinity.

If a point P $(\alpha_1\beta_1\gamma_1)$ be determined by the intersection of the circle ABC with the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$, it may be at once shown that the lines

$$\beta\beta_1-\gamma\gamma_1=o,\quad \gamma\gamma_1-\alpha\alpha_1=o,\quad \alpha\alpha_1-\beta\beta_1=o$$

which determine the position of the inverse of P, are all parallel to the line $l\alpha+m\beta+n\gamma=o$.

A line passing through a vertex of the triangle ABC inverts into a line passing through the same vertex.

3. The conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$ will be a hyperbola, parabola, or ellipse according as

$$\sqrt{la}+\sqrt{mb}+\sqrt{nc}>=\text{or}<o$$

but this is the condition that the line $l\alpha+m\beta+n\gamma=o$ shall intersect, touch, or not intersect the circle ABC: hence the theorem that a line inverts into a hyperbola, parabola, or ellipse according as it cuts, touches, or does not cut the circumcircle of the triangle of reference.

4. The asymptotes of the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$ are given by

$$lmn\,(a\alpha+b\beta+c\gamma)^2+\Delta\,(l\beta\gamma+m\gamma\alpha+n\alpha\beta)=o$$

where

$$\Delta=a^2l^2+b^2m^2+c^2n^2-2bcmn-2canl-2ablm$$

It is easily shown that the angle (ϕ) between the asymptotes is given by

$$\tan\phi=\frac{\sqrt{\Delta}}{2R\,(l\cos A+m\cos B+n\cos C)}$$

R being the radius of the circle ABC. Hence

$$\cos\phi=\frac{l\cos A+m\cos B+n\cos C}{\Omega}$$

where $\Omega^2=l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C$.

If p be the length of the perpendicular from the centre of the circle ABC on $l\alpha+m\beta+n\gamma=o$, then

$$p=\frac{R\,(l\cos A+m\cos B+n\cos C)}{\Omega}$$

therefore $\cos\phi=\frac{p}{R}$: but $\frac{p}{R}$ is the cosine of the angle between the chord $l\alpha+m\beta+n\gamma=o$ of the circle ABC and the tangent to the circle at the extremity of the chord, hence the angle between the asymptotes of the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$ is equal to the angle at which the line $l\alpha+m\beta+n\gamma=o$ cuts the circle ABC.

Moreover, since the excentricity (ϵ) of the conic is connected with ϕ by the relation $\epsilon=\sec\frac{\phi}{2}$ and $p=R\cos\phi$, it follows at once that tangents to circles concentric with the circle ABC invert into similar conics.

5. Suppose that a curve S inverts into a curve S′: then to any two points P and Q on S will be two corresponding inverse points P′ and Q′ on S′. If now the point Q move up to P and become infinitely close to it, the point Q′ will become infinitely close to P′. Hence if the tangent to S at the point P be inverted, it will become a circumconic touching S′ at the point P′.

If the line $l\alpha+m\beta+n\gamma=o$ be inverted, then any tangent to the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$ will invert into a conic touching $l\alpha+m\beta+n\gamma=o$, and a pair of tangents to the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$ will invert into a pair of circumconics intersecting in the point which is the inverse of that from which the tangents were drawn and having the line $l\alpha+m\beta+n\gamma=o$ as a common tangent.

6. Let two lines $L_1=l_1\alpha+m_1\beta+n_1\gamma=o$ and $L_2=l_2\alpha+m_2\beta+n_2\gamma=o$ be taken: these will invert into the conics

$$S_1=l_1\beta\gamma+m_1\gamma\alpha+n_1\alpha\beta=o$$
$$S_2=l_2\beta\gamma+m_2\gamma\alpha+n_2\alpha\beta=o$$

Let $L=\lambda\alpha+\mu\beta+\nu\gamma=o$ be a common tangent of S_1 and S_2: then L will invert into the conic

$$S=\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=o$$

which will have double contact with the line pair $L_1\,L_2$: its equation therefore will be of the form

$$L_1L_2-(p\alpha+q\beta+r)^2=o$$

Comparing this with the form of S given above we have

$$p^2 = l_1 l_2,\ q^2 = m_1 m_2,\ r^2 = n_1 n_2$$

hence the equations of the four chords of contact with L_1 and L_2 of the conics which are the inverses of the common tangents of S_1 and S_2 are

$$\sqrt{l_1 l_2}\,\alpha \pm \sqrt{m_1 m_2}\,\beta + \sqrt{n_1 n_2}\,\gamma = o$$

The inverses of the points in which these four lines meet L_1 and L_2 are the points of contact of the common tangents of S_1 and S_2.

$$\begin{aligned}
\text{Let } c_1 &= \sqrt{l_1 l_2}\,\alpha + \sqrt{m_1 m_2}\,\beta + \sqrt{n_1 n_2}\,\gamma = o \\
c_2 &= \sqrt{l_1 l_2}\,\alpha - \sqrt{m_1 m_2}\,\beta - \sqrt{n_1 n_2}\,\gamma = o \\
c_3 &= -\sqrt{l_1 l_2}\,\alpha + \sqrt{m_1 m_2}\,\beta - \sqrt{n_1 n_2}\,\gamma = o \\
c_4 &= -\sqrt{l_1 l_2}\,\alpha - \sqrt{m_1 m_2}\,\beta + \sqrt{n_1 n_2}\,\gamma = o
\end{aligned}$$

and form the conic

$$T_1 = L_1 L_2 - c_1^2 = o$$

which is the inverse of a common tangent t_1.

Now write

$$\begin{aligned}
P_1 &= \sqrt{m_1 n_2} - \sqrt{m_2 n_1} & P_2 &= \sqrt{m_1 n_2} + \sqrt{m_2 n_1} \\
Q_1 &= \sqrt{n_1 l_2} - \sqrt{n_2 l_1} & Q_2 &= \sqrt{n_1 l_2} + \sqrt{n_2 l_1} \\
R_1 &= \sqrt{l_1 m_2} - \sqrt{l_2 m_1} & R_2 &= \sqrt{l_1 m_2} + \sqrt{l_2 m_1}.
\end{aligned}$$

Then the conic T_1 reduces to

$$P_1^2 \beta\gamma + Q_1^2 \gamma\alpha + R_1^2 \alpha\beta = o$$

On inversion we obtain the four common tangents of S_1 and S_2

$$\begin{aligned}
t_1 &= P_1^2\alpha + Q_1^2\beta + R_1^2\gamma = o \\
t_2 &= P_1^2\alpha + Q_2^2\beta + R_2^2\gamma = o \\
t_3 &= P_2^2\alpha + Q_1^2\beta + R_2^2\gamma = o \\
t_4 &= P_2^2\alpha + Q_2^2\beta + R_1^2\gamma = o
\end{aligned}$$

To find the co-ordinates of the points of contact of t_1 with S_1 and S_2, solve for $\alpha\beta\gamma$ between c_1 and L_1 and c_1 and L_2 and invert.

We thus find that t_1 will touch S_1 and S_2 respectively in the points

$$\left(\frac{\sqrt{l_1}}{P_1}\ \frac{\sqrt{m_1}}{Q_1}\ \frac{\sqrt{n_1}}{R_1}\right) \qquad \left(\frac{\sqrt{l_2}}{P_1}\ \frac{\sqrt{m_2}}{Q_1}\ \frac{\sqrt{n_2}}{R_1}\right)$$

with similar expressions for the points of contact of t_2, t_3, and t_4 with these conics.

7. Any triangle circumscribing the conic

$$S = \lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta = o$$

will invert into three circumconics having the line

$$L = \lambda\alpha + \mu\beta + \nu\gamma = o$$

as a common tangent.

A family of n parabolas circumscribing the triangle of reference will invert into an n-sided polygon in which the circle ABC is inscribed.

The pencil of lines represented by the equation

$$l_1\alpha + m_1\beta + n_1\gamma + \kappa\,(l_2\alpha + m_2\beta + n_2\gamma) = o$$

where κ varies, will invert into a family of conics passing through the four points of intersection of the conics

$$l_1\beta\gamma + m_1\gamma\alpha + n_1\alpha\beta = o$$
$$l_2\beta\gamma + m_2\gamma\alpha + n_2\alpha\beta = o$$

In particular a system of parallel lines will invert into a family of conics passing through four concyclic points.

Hence, as there will always be two lines, whether of the pencil or of the parallel system, which are equidistant from the centre of the circle ABC (excluding those lines of either system which are diameters of this circle), we see that all conics passing through four given points may be arranged in pairs of similar conics.

8. Two tangents drawn from a point P to the circle ABC will invert into two parabolas passing through ABC and P′—the inverse of P with respect to the triangle ABC.

Hence if four points, ABCD, be given, and if A′ B′ C′ D′ be respectively the inverses of those points with respect to the triangle formed by joining the remaining three points, we see that the two parabolas which may be drawn through four given points can be regarded as originating by inversion of the pair of tangents from the four points A′ B′ C′ D′ to the circles BCD, CDA, DAB, ABC respectively.

Now, if one of the points, say D′, fall within the circle ABC, the tangents from it to that circle are imaginary; and consequently the two parabolas through ABCD are imaginary: therefore the remaining points A′ B′ C′ must lie within the respective circles BCD, CDA, DAB.

We may state this result as follows: If any four points be taken on a parabola, the inverse of any one of the points with respect to the triangle formed by joining the remaining three points lies without the circumcircle of that triangle.

9. We may determine the equation of the two parabolas which can be drawn through ABC and $P(\alpha_1\beta_1\gamma_1)$ as follows:—

The curve whose equation is

$$\sqrt{\frac{a}{\alpha}}+\sqrt{\frac{b}{\beta}}+\sqrt{\frac{c}{\gamma}}=o$$

is the locus of points whose axes of homology touch the circle ABC, while the conic

$$\frac{1}{\alpha_1\alpha}+\frac{1}{\beta_1\beta}+\frac{1}{\gamma_1\gamma}=o$$

is the locus of points whose axes of homology pass through

$$\left(\frac{1}{\alpha_1}\ \frac{1}{\beta_1}\ \frac{1}{\gamma_1}\right)$$

Let these two curves cut in the point $\alpha'\beta'\gamma'$: then

$$\frac{\alpha}{\alpha'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=o$$

will be a tangent to the circle through $\frac{1}{\alpha_1}\ \frac{1}{\beta_1}\ \frac{1}{\gamma_1}$

We have also

$$\frac{1}{\alpha\alpha'}+\frac{1}{\beta\beta'}+\frac{1}{\gamma\gamma'}=o$$

$$\sqrt{\frac{a}{\alpha'}}+\sqrt{\frac{b}{\beta'}}+\sqrt{\frac{c}{\gamma'}}=o$$

whence, eliminating $\alpha'\beta'\gamma'$, we have the equation of the two tangents in the form

$$\sqrt{a\alpha_1(\beta_1\beta-\gamma_1\gamma)}+\sqrt{b\beta_1(\gamma_1\gamma-\alpha_1\alpha)}+\sqrt{c\gamma_1(\alpha_1\alpha-\beta_1\beta)}=o$$

and the equation of the pair of parabolas is

$$\sqrt{a\alpha\left(\frac{\beta}{\beta_1}-\frac{\gamma}{\gamma_1}\right)}+\sqrt{b\beta\left(\frac{\gamma}{\gamma_1}-\frac{\alpha}{\alpha_1}\right)}+\sqrt{c\gamma\left(\frac{\alpha}{\alpha_1}-\frac{\beta}{\beta_1}\right)}=o$$

10. Let there be four concyclic points A, B, C, D, and let the position of the point D be determined by the intersection of the circle ABC and the conic

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=o$$

Then the two parabolas through the four points will be the inverses of the two tangents to the circle ABC which are parallel to the line $l\alpha+m\beta+n\gamma=o$.

Consider the conic whose equation is

$$\frac{mc-nb}{\alpha}+\frac{na-lc}{\beta}+\frac{lb-ma}{\gamma}=$$

It is the locus of points whose axes of homology are parallel to $l\alpha+m\beta+n\gamma = o$.

Let this conic cut in the point $(\alpha'\beta'\gamma')$ the curve

$$\sqrt{\frac{a}{\alpha}}+\sqrt{\frac{b}{\beta}}+\sqrt{\frac{c}{\gamma}} = o$$

Then the axis of homology of $(\alpha'\beta'\gamma')$ will be a tangent to the circle ABC and parallel to the line $l\alpha+m\beta+n\gamma = o$. Eliminating $\alpha'\beta'\gamma'$ between the equations

$$\frac{\alpha}{\alpha'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} = o$$

$$\frac{mc-nb}{\alpha'}+\frac{na-lc}{\beta'}+\frac{lb-ma}{\gamma'} = o$$

$$\sqrt{\frac{a}{\alpha'}}+\sqrt{\frac{b}{\beta'}}+\sqrt{\frac{c}{\gamma'}} = o$$

we have for the equation of the pair of tangents

$$a\sqrt{\frac{l}{a}(b\beta+c\gamma)-(m\beta+n\gamma)}+b\sqrt{\frac{m}{b}(c\gamma+a\alpha)-(n\gamma+l\alpha)}$$
$$+c\sqrt{\frac{n}{c}(a\alpha+b\beta)-(l\alpha+m\beta)} = o$$

The equation of the two parabolas may be written down from the above by substituting in it $\frac{1}{\alpha}\,\frac{1}{\beta}\,\frac{1}{\gamma}$ for $\alpha\ \beta\ \gamma$ respectively.

11. Any line parallel to $\alpha = o$ will invert into a conic of the form

$$\kappa\beta\gamma+a\beta\gamma+b\gamma\alpha+c\alpha\beta = o$$

All conics of this family touch each other and the circle ABC at the vertex A of the triangle of reference.

The two tangents to the circle ABC parallel to $\alpha = o$ invert into the pair of parabolas

$$(b\pm c)^2\beta\gamma+a\alpha(b\gamma+c\beta) = o$$

The two tangents to the same circle drawn parallel to the diameter of the circle through A invert into the pair of parabolas

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta\pm 4\,\mathrm{R}\sin\mathrm{B}\sin\mathrm{C}\,\alpha\,(\beta\cos\mathrm{B}-\gamma\cos\mathrm{C}) = o$$

where R is the radius of the circle ABC.

12. Any diameter of the circle ABC will invert into a rectangular hyperbola.

Let the diameter be taken which is perpendicular to the line $\alpha = o$; the equation of its inverse is

$$\frac{\sin(B-C)}{\alpha} + \frac{\sin B}{\beta} - \frac{\sin C}{\gamma} = o$$

This conic cuts the circle ABC at the extremity H of the diameter which passes through the vertex A of the triangle of reference; it also passes through the points $\left(-\frac{1}{a}\,\frac{1}{b}\,\frac{1}{c}\right)$ and the orthocentre of the triangle ABC: its centre is at the middle point of the line BC: the tangent to the conic at A passes through the symmedian point (*abc*) of the $\triangle$ ABC, while the tangent at H passes through the point (−*abc*).

Art. XXX.—*Some Observations on the Coastal Vegetation of the South Island of New Zealand.—Part I: General Remarks on the Coastal Plant Covering.*

By L. Cockayne, Ph.D.

[*Read before the Philosophical Institute of Canterbury, 8th August,* 1906.]

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